

# Entanglement Measure of the Planar-Transverse Classical Light Field

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Classical light fields are considered physical examples of nonquantum entanglement [Qian and Eberly, Opt. Lett. **36**, 4110 (2011)]. We apply concurrence and the Schmidt approach to evaluate the degree of entanglement for a generalized polarization state that Qian and Eberly suggested, and we obtained an analytic form of the general entanglement state for planar polarization states of classical light. Calculating the concurrence and the Schmidt value for the usual polarization states in two-dimensional lab space, we found a maximum entanglement state for the two-dimensional lab space unit vectors ( $\vec{x}$  and the  $\vec{y}$ ) and two-dimensional statistical function space unit vectors ( $E_x$  and  $E_y$ ).

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## I. INTRODUCTION

Entanglement was considered a purely quantum mechanical property until Eberly suggested physical examples of nonquantum entanglement [1,2]. They identified the polarization of classical light fields as a physical example of nonquantum entanglement. After Stokes defined the polarization state [3], the formulation of the polarization state has continuously evolved and is now well established in terms of field correlation functions [4]. The familiar measures of polarization come from the paraxial field approximation. But, the traditional picture of polarization must be reconsidered for fully three-dimensional fields such as highly nonparaxial fields [5–8]

Those studies extended the electric field from planar-transverse to nonplanar such as

$$\vec{E} = \hat{x}E_x + \hat{y}E_y \Rightarrow \hat{x}E_x + \hat{y}E_y + \hat{z}E_z, \quad (1)$$

and employed two independent vector spaces that are entangled to realize  $\vec{E}$  in Eq. (1). Entanglement is a technical term indicated by  $\vec{E}$  in Eq. (1). It is a tensor product of “lab space” unit vectors, such as  $\vec{x}$  and  $\vec{y}$ , and

functions  $E_x$  and  $E_y$ , which are vectors in a statistical “function space” of continuous normed functions.

In a mathematical sense, determining the degree of polarization is the same as the determining degree of factorization of two spaces. In this work, we obtained an complete form of the general entanglement state for planar polarization states of classical light.

## II. SCHMIDT VALUE AND CONCURRENCE

### 1. Schmidt value

Two independent vector spaces are introduced to express the electric field.  $\vec{E}$  is a tensor product of “lab space” unit vectors, such as  $\vec{x}$  and  $\vec{y}$ , and functions  $E_x$  and  $E_y$ , which are vectors in a statistical “function space” of continuous normed functions [1]. With intensity  $I = \langle E_x | E_x \rangle + \langle E_y | E_y \rangle$  factored out,

$$|\vec{E}\rangle / \sqrt{I} = (\cos\theta | \vec{x} \rangle | e_x \rangle + \sin\theta | \vec{y} \rangle | e_y \rangle), \quad (2)$$

the relative amplitudes via the sine and cosine factors allows the components  $|e_x\rangle, |e_y\rangle$  to be unit normalized. The nonzero correlation is included between field

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components by introducing the magnitude and phase of the correlation as  $\langle e_x | e_y \rangle \equiv \alpha$ . The electric field in Eq. (2) can be written as [1]

$$|\vec{E}\rangle / \sqrt{I} = (\cos\theta|\vec{x}\rangle + \alpha\sin\theta|\vec{y}\rangle)|e_x\rangle + \beta\sin\theta|\vec{y}\rangle|\bar{e}_x\rangle, \quad (3)$$

with  $|\bar{e}_x\rangle$  as an orthogonal components of  $|e_x\rangle$ , such that  $\langle e_x | \bar{e}_x \rangle = 0$  and  $|e_y\rangle = \alpha|e_x\rangle + \beta|\bar{e}_x\rangle$ , where  $\alpha$  and  $\beta$  satisfy the relation  $|\alpha|^2 + |\beta|^2 = 1$ .

Applying the Schmidt analysis [9] for the state  $|\vec{E}\rangle / \sqrt{I}$  in Eq. (3), the original state can be written as

$$|\vec{E}\rangle / \sqrt{I} = \frac{\sqrt{\lambda_1}}{\sqrt{\eta_1^2 + 4|\alpha|^2\sqrt{\zeta_1^2 + 4|\alpha\beta|^2}}}(\eta_1|\vec{x}\rangle + 2\alpha|\vec{y}\rangle) \otimes (\zeta_1|e_x\rangle + 2\alpha\beta^*|\bar{e}_x\rangle) + \frac{\sqrt{\lambda_2}}{\sqrt{\eta_2^2 + 4|\alpha|^2\sqrt{\zeta_2^2 + 4|\alpha\beta|^2}}}(\eta_2|\vec{x}\rangle + 2\alpha|\vec{y}\rangle) \otimes (\zeta_2|e_x\rangle + 2\alpha\beta^*|\bar{e}_x\rangle), \quad (4)$$

where

$$\eta_1 = \cot\theta - \csc\theta \sec\theta \sqrt{1 - |\beta|^2 \sin^2 2\theta} - \tan\theta \quad (5)$$

$$\eta_2 = \cot\theta + \csc\theta \sec\theta \sqrt{1 - |\beta|^2 \sin^2 2\theta} - \tan\theta \quad (6)$$

$$\zeta_1 = 1 - 2|\beta|^2 + \cot^2\theta - \csc^2\theta \sqrt{1 - |\beta|^2 \sin^2 2\theta} \quad (7)$$

$$\zeta_2 = 1 - 2|\beta|^2 + \cot^2\theta + \csc^2\theta \sqrt{1 - |\beta|^2 \sin^2 2\theta}. \quad (8)$$

and the eigenvalues are

$$\lambda_1 = \frac{1}{2}(1 - \sqrt{1 - |\beta|^2 \sin^2 2\theta}) \quad (9)$$

$$\lambda_2 = \frac{1}{2}(1 + \sqrt{1 - |\beta|^2 \sin^2 2\theta}). \quad (10)$$

The Schmidt number as the degree of entanglement can be easily computed from the eigenvalues  $\lambda_1$  and  $\lambda_2$

$$K = \frac{1}{\sum_s \lambda_s^2} = \frac{1}{1 - \frac{1}{2}|\beta|^2 \sin^2 2\theta}. \quad (11)$$

We plotted the degree of entanglement  $K$  with respect to  $\alpha$  ( $= \sqrt{1 - |\beta|^2}$ ) and the angle  $\theta$  in Fig. 1. The degree of entanglement  $K$  may have a maximum of 2 under some conditions such as  $\beta = 1$  and  $\theta = \frac{\pi}{4}$ . Under this condition, the electric field  $E$  in Eq. (4) becomes

$$|\vec{E}\rangle_{max} / \sqrt{I} = \frac{1}{\sqrt{2}}|\vec{x}\rangle \otimes |e_x\rangle + \frac{1}{\sqrt{2}}|\vec{y}\rangle \otimes |\bar{e}_x\rangle \quad (12)$$

Mathematically the electric fields  $E_x$  and  $E_y$  might create vectors in function space [1], and  $|\bar{e}_x\rangle$  is orthogonal to  $|e_x\rangle$ . However, it is still unclear how to make  $|\vec{E}\rangle_{max}$  in Eq. (12).

In contrast, it is easy to make a state in which the degree of entanglement is a minimum value of 1. The degree of entanglement  $K$  may have a minimum of 1 when  $\beta = 0$ . In this case, the electric field  $E$  in Eq. (4) becomes

$$|\vec{E}\rangle_{m1} / \sqrt{I} = (\cos\theta|\vec{x}\rangle + \sin\theta|\vec{y}\rangle) \otimes |e_x\rangle. \quad (13)$$

This state is simply the linearly polarized state with single component in function space. We also obtain the minimum  $K$  value 1 by  $\theta = \frac{\pi}{2}$ .

$$|\vec{E}\rangle_{m2} / \sqrt{I} = |\vec{y}\rangle \otimes (\alpha|e_x\rangle + \beta|\bar{e}_x\rangle), \quad (14)$$

At this time, the  $|\vec{E}\rangle_{m2}$  state has a single component in “lab space”.

## 2. Concurrence

For the electric field  $E$  in Eq. (4), we can obtain the density matrix,  $\rho$ , based on  $\{|x e_x\rangle, |x \bar{e}_x\rangle, |y e_x\rangle,$

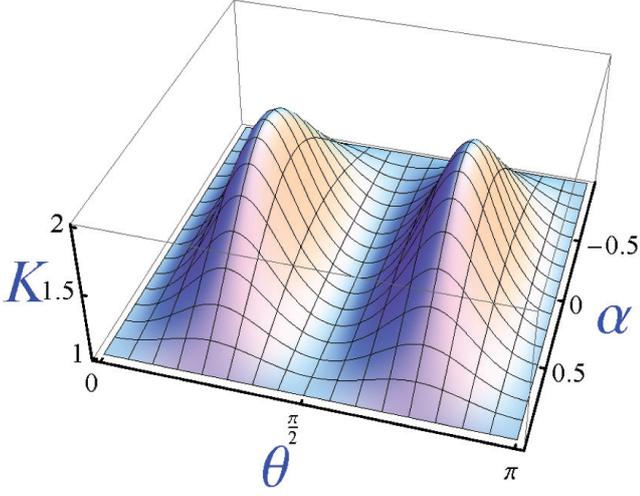


Fig. 1. (Color online) Degree of entanglement  $K$  with respect to  $\alpha$  and the angle  $\theta$ .

$|y \bar{e}_x \rangle\}$ ,

$$\rho_A = \begin{pmatrix} \cos^2 \theta & 0 & \alpha^* \cos \theta \sin \theta & \beta^* \cos \theta \sin \theta \\ 0 & 0 & 0 & 0 \\ \alpha \cos \theta \sin \theta & 0 & |\alpha|^2 \sin^2 \theta & \alpha \beta^* \sin^2 \theta \\ \beta \cos \theta \sin \theta & 0 & \alpha^* \beta \sin^2 \theta & |\beta|^2 \sin^2 \theta \end{pmatrix} \quad (15)$$

We can calculate the concurrence of this system from this density matrix. The explicit formula for concurrence  $C(\rho)$  is: [10]

$$C(\rho) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}, \quad (16)$$

where  $\lambda_i$  is eigenvalue of  $\rho \bar{\rho}$  in descending order. Here  $\bar{\rho}$  is the result of applying the Pauli operator to  $\rho$ ,

$$\bar{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y). \quad (17)$$

We obtain the concurrence from the density matrix  $\rho_A$ ,

$$C(\rho_A) = \frac{1}{2} |(\beta + \beta^*) \sin 2\theta|. \quad (18)$$

We plotted the concurrence  $C$  with respect to  $\alpha (= \sqrt{1 - |\beta|^2})$  and the angle  $\theta$  in Fig. 2. We plotted the maximum concurrence for a given  $|\beta|$  with  $Im(\beta) = 0$ , as  $\beta + \beta^*$  has its maximum if  $\beta$  is a real number.

The concurrence,  $C$ , is maximum when  $\alpha = 0$  and  $\theta = \frac{\pi}{4}$ . Under this condition, the electric field can be written as in Eq. (12). The minimum value of the concurrence can be obtained with  $\theta = \frac{\pi}{2}$  or with  $\beta = 0$  as in Eqs. (13) and (14).

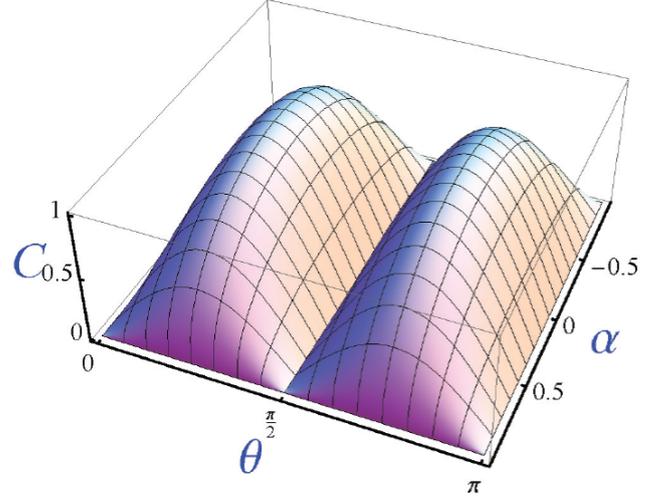


Fig. 2. (Color online) The concurrence,  $C$ , with respect to  $\alpha$  and the angle  $\theta$ .

### III. CONCLUSION AND DISCUSSION

Classical light fields may be considered as physical examples of nonquantum entanglement. Qian and Eberly reformulated polarization theory as entanglement analysis. In this perspective, polarization is a characterization of the correlation between the vector nature and the statistical nature of the light field. Those authors discussed the general entanglement Schmidt value  $K$ , which varies from 1 to 3 over the unit polarization sphere for nonplanar case.

We applied the concurrence and Schmidt approach to evaluate the degree of entanglement for a generalized polarization state for the planar-transverse case. We found a maximum entanglement state for the two-dimensional lab space unit vector ( $\hat{x}$  and  $\hat{y}$ ) and two-dimensional statistical function space unit vectors ( $E_x$  and  $E_y$ ).

Although, it is not clear how to measure the degree of entanglement over lab and function space, we calculated the concurrence and the Schmidt value for the usual polarization states in two-dimensional lab space. We expect some experiments to measure the degree of entanglement of the maximally entangled polarization state.

### ACKNOWLEDGEMENTS

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