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Landau States in Time-Dependent Magnetic Fields

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We present the oscillator representation of the Pauli Hamiltonian for a scalar charge in a magnetic field and find a basis that diagonalizes the Hamiltonian in the special case of a constant or slowly varying magnetic field. We show that the diagonalization of the new basis is a canonical transformation in phase space, which leads to the Pauli Hamiltonian and counts the degeneracy of the Landau levels. Finally, we obtain the Liouville-von Neumann equation for quantum invariants as the annihilation and the creation operators for a scalar charge in time-dependent magnetic fields.

Keywords: Landau levels, Time-dependent magnetic field, Quantum invariants, Time-dependent annihilation and creation operators

I. INTRODUCTION

The interaction of a charge with electromagnetic fields is important in understanding the electromagnetic property of condensed matter, the equation of state and the quantum vacuum structure. The interest in Landau levels in strong magnetic fields has been brought by the equation of states in neutron stars, whose magnetic fields are stronger than $B_a = 2.35 \times 10^9\, G$ in which the Landau level energy equals to the Bohr atomic energy: $\hbar\omega_c = \hbar(eB_a/mc) = e^2/a_B$ with the Bohr radius $a_B = \lambda_C/\alpha$ ($\lambda_C = \hbar/mc, \alpha = e^2/\hbar c$). A strong magnetic field ($B \geq B_a$) drastically changes the equation of state of matter in neutron stars by forcing electrons into the lowest Landau level instead of the atomic ground state [1]. On the other hand, the magnetic field whose Landau level spacing equals to the rest mass of the electron gives the Schwinger field $B_c = m^2c^3/\hbar e = 4.41 \times 10^{13}\, G$. Any magnetic field of $B \geq B_c$ will change the quantum structure of the Dirac vacuum and polarize the vacuum [2]. The energy density of the Schwinger field equals the rest mass of the electron. Note that $B_a = \alpha^2 B_c$.

In relativistic theory the Landau levels are found by solving the Dirac or Klein-Gordon equation for a charge in magnetic fields. Then, the exact one-loop effective action is the Heisenberg-Euler and Schwinger QED action [3,4]. The vacuum polarization, the real part of one-loop effective action, results in different refraction indices for parallel and perpendicular propagation modes, the vacuum birefringence. Strong electric fields create pairs of charges, the so-called Schwinger pair production, which is a consequence of the vacuum persistence, the imaginary part of one-loop effective action. The Schwinger pair production exhibits the Stokes phenomenon for bosons [5] and fermions [6] (references therein). Recently the gamma-function regularization has been introduced to find the QED action in the Sauter-type electric and magnetic fields [7,8]. The Landau levels of a scalar charge are nontrivial in time-dependent magnetic fields [9]. The functional Schrödinger picture has been employed to formulate the second quantized Klein-Gordon equation in time-dependent electromagnetic fields [10].

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In this paper, we study the Landau states of a scalar charge in time-dependent magnetic fields in nonrelativistic theory. The electromagnetic property of matter is prescribed by the Pauli Hamiltonian for electrons interacting with electromagnetic fields and constituents. A scalar charge in an electromagnetic field is given by the Pauli Hamiltonian for the time-dependent Schrödinger equation (in unit of $\hbar = c = 1$)

$$i\frac{\partial}{\partial t}|\Psi\rangle = \Big[\frac{1}{2m}\big(\hat{\mathbf{p}} - q\mathbf{A}\big)^2 + q\phi\Big]|\Psi\rangle, \tag{1}$$

where ϕ is the Coulomb potential and $\mathbf{A}$ is the vector potential for the electromagnetic field. Here, one may add the spin interaction $(q/2m)\sigma\cdot\mathbf{B}$ for electrons. Laughlin obtained the wave function for electrons in a magnetic field and explained the quantum Hall effect [11]. A charge in a constant magnetic field has been used as a model for planar physics (for review, see Ref. [12]). The massless QED has been discussed in graphene and Dirac semimetals [13]. The Landau states of a scalar charge in time-dependent magnetic fields have been studied by the evolution operators and unitary transformations [14–16] and the invariant operator approach [17,18]. The invariant operator approach was applied to coupled oscillators [19,20].

The organization of this paper is as follows. In Sec. II we review the quantum invariants for time-dependent oscillators and find the coherent-number states. In Sec. III we use the oscillator representation for a scalar charge in constant or time-dependent magnetic fields and find the Landau levels and the degeneracy. In Sec. IV we introduce the linear quantum invariants as the time-dependent annihilation and creation operators for the planar Hamiltonian in time-dependent magnetic fields.

II. Invariants for Time-Dependent Harmonic Oscillator

Lewis and Riesenfeld introduced a quadratic quantum invariant [21]

$$\hat{I}(t) = \frac{1}{2}\Big[\big(\xi\hat{p} - \dot{\xi}\hat{q}\big)^2 + \Big(\frac{\hat{q}}{\xi}\Big)^2\Big], \tag{2}$$

for the time-dependent oscillator

$$\hat{H}(t) = \frac{\hat{p}^2}{2m} + \frac{m}{2}\omega^2(t)\hat{q}^2. \tag{3}$$

Here, ξ is the solution to the Ermakov system [22,23]

$$\ddot{\xi} + \omega^2(t)\xi = \frac{1}{4\xi^3}. \tag{4}$$

Then, the exact wave function of time-dependent Schrödinger equation is given by

$$|\Psi(t)\rangle = \sum_n C_n e^{i\int\langle n,t|i\frac{\partial}{\partial t} - \hat{H}_0(t)|n,t\rangle}|n,t\rangle \tag{5}$$

where C_n are constants and

$$\hat{I}(t)|n,t\rangle = \lambda_n|n,t\rangle. \tag{6}$$

Here, λ_n are constants of motion in general and $\lambda_n = n + 1/2$ for the invariant (2). The Hamiltonian $\hat{H}$ has two linear invariants, which can be interpreted as the time-dependent annihilation and creation operators [24,25]

$$\begin{aligned}\hat{a}(t) &= i(w^*\hat{p} - m\dot{w}^*\hat{q}),\\ \hat{a}^\dagger(t) &= -i(w\hat{p} - m\dot{w}\hat{q}).\end{aligned} \tag{7}$$

where w is a complex solution to

$$\ddot{w}(t) + \omega^2(t)w(t) = 0, \tag{8}$$

and satisfies the Wronskian condition

$$m\mathrm{Wr}[w, w^*] = i. \tag{9}$$

The magnitude ξ of the complex solution of Eqs. (8) and (9) in the form

$$w(t) = \frac{\xi(t)}{\sqrt{m}}e^{-i\int\frac{1}{2\xi^2}} \tag{10}$$

is the solution of Eq. (4).

The number states are

$$|n,t\rangle = \frac{\big(\hat{a}^\dagger(t)\big)^n}{\sqrt{n!}}|0,t\rangle, \tag{11}$$

where the ground state is defined by

$$\hat{a}(t)|0,t\rangle = 0. \tag{12}$$

Then, the coordinate-representation of a coherent-number state is [26]

$$\begin{aligned}\Psi_n(q,\xi) &= \sqrt{\frac{\sqrt{m}}{\sqrt{2\pi}2^n n!\xi}}e^{-i\left(n+\frac{1}{2}\right)\int\frac{1}{2\xi^2}}e^{ip_c(t)q}\\ &\times H_n\Big(\frac{\sqrt{m}(q-q_c(t))}{\sqrt{2}\xi}\Big)e^{-\frac{m}{2}\left(\frac{1}{\xi^2} - i\frac{\dot{\xi}}{\xi}\right)\left(q-q_c(t)\right)^2},\end{aligned} \tag{13}$$

where $q_c(t)$ and $p_c(t)$ are the classical position and momentum

$$q_c(t) = \langle \zeta, t|\hat{q}|\zeta, t\rangle, \quad p_c(t) = \langle \zeta, t|\hat{p}|\zeta, t\rangle, \tag{14}$$

where $|\zeta, t\rangle$ is a coherent state:

$$\hat{a}(t)|\zeta, t\rangle = \zeta|\zeta, t\rangle \tag{15}$$

for a complex number ζ. The squeezed state and thermal state are straightforwardly found by the annihilation and creation operators (7) [26–28].

III. Oscillator Representation for Scalar Charge in Magnetic Fields

The planar Hamiltonian for a scalar charge in the symmetric gauge ($\mathbf{A} = -\mathbf{r} \times \mathbf{B}/2$) of magnetic field is

$$\hat{H}_\perp = \sum_{i=1,2} \frac{1}{2m}\hat{p}_i^2 + \frac{m}{2}\omega^2(t)\hat{x}_i^2 - \omega(t)\hat{L}_3, \tag{16}$$

where $\hat{L}_3 = \hat{x}_1\hat{p}_2 - \hat{p}_1\hat{x}_2$ and

$$\omega(t) = \frac{qB(t)}{2m} = \frac{\omega_c(t)}{2}. \tag{17}$$

One may introduce an annihilation operator and a creation operator

$$\begin{aligned} \hat{x}_i &= \Big(\frac{1}{2m\omega_0}\Big)^{1/2}\big(\hat{a}_i^\dagger + \hat{a}_i\big), \\ \hat{p}_i &= i\Big(\frac{m\omega_0}{2}\Big)^{1/2}\big(\hat{a}_i^\dagger - \hat{a}_i\big). \end{aligned} \tag{18}$$

Then the planar Hamiltonian has the oscillator representation

$$\begin{aligned} \hat{H}_\perp &= \frac{1}{2}\sum_{i\neq j}^{1,2}\Big[\frac{\Omega_+(t)}{2}\Big(\hat{a}_i^\dagger\hat{a}_i + \frac{1}{2}\Big) + \frac{\Omega_-(t)}{2}\Big(\hat{a}_i^{\dagger 2} + \hat{a}_i^2\Big) \\ &\quad +2i\omega(t)\epsilon_{ij}\hat{a}_i^\dagger\hat{a}_j\Big], \end{aligned} \tag{19}$$

where

$$\Omega_\pm(t) = \frac{\omega^2(t)}{\omega_0} \pm \omega_0. \tag{20}$$

Here, ϵ_{ij} is the Kronecker epsilon, and the second summation is a one-mode squeezing operator while the third summation is a two-mode squeezing operator.

The number states for $\hat{a}_i$ are not the eigenstates of the oscillator representation (19) because of the squeezing terms, the second and the third summations. However, a new basis

$$\hat{a}_1 = \frac{1}{\sqrt{2}}\big(\hat{b} + \hat{c}\big), \quad a_2 = \frac{i}{\sqrt{2}}\big(\hat{b} - \hat{c}\big), \tag{21}$$

leads to another oscillator representation

$$\begin{aligned} \hat{H}_\perp &= \frac{1}{2}\Big[\Omega_-(t)\big(\hat{b}^\dagger\hat{c}^\dagger + \hat{b}\hat{c}\big) + \Omega_+(t)\big(\hat{b}^\dagger\hat{b} + \hat{c}^\dagger\hat{c} + 1\big) \\ &\quad +2\omega(t)\big(\hat{c}^\dagger\hat{c} - \hat{b}^\dagger\hat{b}\big)\Big]. \end{aligned} \tag{22}$$

Hence, a static Hamiltonian in a constant magnetic field has a diagonal form by choosing $\omega_0 = \omega$. It also follows that $\omega_0 = \omega(t)$ for a time-dependent magnetic field gives an instantaneous diagonalization

$$\hat{H}_\perp(t) = \omega(t)\big(2\hat{c}^\dagger(t)\hat{c}(t) + 1\big). \tag{23}$$

It should be stressed that the oscillator representation (23) does not provide the exact quantum states for a charge in time-dependent magnetic fields.

In order to understand the degeneracy of the Landau levels, we first consider the Landau levels in a constant magnetic field. The Hamiltonian and the angular momentum have the representation, respectively,

$$\hat{H}_\perp = \omega_0\big(2\hat{c}^\dagger\hat{c} + 1\big), \tag{24}$$

and

$$\hat{L}_3 = \big(\hat{b}^\dagger\hat{b} - \hat{c}^\dagger\hat{c}\big). \tag{25}$$

The planar Hamiltonian has the two annihilation operators for the ground state

$$\hat{c}|0_c, 0_b\rangle = \hat{b}|0_c, 0_b\rangle = 0. \tag{26}$$

Thus the excited states are given by

$$|n, l\rangle = \frac{\big(\hat{c}^\dagger\big)^n}{\sqrt{n!}}\frac{\big(\hat{b}^\dagger\big)^l}{\sqrt{l!}}|0, 0\rangle. \tag{27}$$

Hence, the Landau level with an energy $H_\perp = \omega_0(2n+1)$ and an angular momentum $L_3 = l$ is

$$|n, n + l\rangle, \quad (n, l = 0, 1, 2\cdots). \tag{28}$$

Note that the lowest Landau level has an infinite degeneracy

$$|0, l\rangle, \quad (l = 0, 1, 2, \cdots). \tag{29}$$

The Landau levels are exact for the static Hamiltonian while they are adiabatic states when the magnetic field slowly varies.

We find the canonical transformation that leads to the Hamiltonian (23),

$$\begin{pmatrix} \sqrt{m\omega_0}\hat{x}_1 \\ \hat{p}_1/\sqrt{m\omega_0} \\ \hat{p}_2/\sqrt{m\omega_0} \\ \sqrt{m\omega_0}\hat{x}_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} I_{2\times 2} & \sigma_1 \\ \sigma_3 & -i\sigma_2 \end{pmatrix} \begin{pmatrix} \sqrt{m\omega_0}\hat{q}_b \\ \hat{p}_b/\sqrt{m\omega_0} \\ \hat{p}_c/\sqrt{m\omega_0} \\ \sqrt{m\omega_0}\hat{q}_c \end{pmatrix}. \tag{30}$$

Here, σ_i are Pauli spin matrices and $I_{2\times 2}$ is the identity matrix. The new phase-space variables satisfy

$$[\hat{q}_b, \hat{p}_b] = i, \quad [\hat{q}_c, \hat{p}_c] = i. \tag{31}$$

The canonical transformation is an orthogonal transformation $\mathcal{M}^T\mathcal{M} = \mathcal{M}\mathcal{M}^T = I_{4\times 4}$ with the unit determinant $\det(\mathcal{M}) = 1$ while $\mathrm{Tr}(\mathcal{M}) = \sqrt{2}$. The planar Hamiltonian in the new phase-space has the form of an oscillator for q_c

$$\begin{aligned} \hat{H}_c &\equiv \frac{\hat{p}_c^2}{2(m/2)} + \frac{(m/2)}{2}(2\omega)^2\hat{q}_c^2 \\ &= \frac{1}{2m}\left[\left(\hat{p}_1 + m\omega\hat{x}_2\right)^2 + \left(\hat{p}_2 - m\omega\hat{x}_1\right)^2\right]. \end{aligned} \tag{32}$$

IV. Invariants for Scalar Charge in Time-Dependent Magnetic Fields

In this section we study the Landau Levels in time-dependent magnetic fields $B(t)$. As explained in Sec. III, the Pauli Hamiltonian in time-dependent magnetic fields consists of coupled oscillators through the angular momentum. When the magnetic field changes slowly, the adiabatic theorem gives the instantaneous Landau levels as mentioned before.

Our stratagem here is to find the linear invariant operators that play the role of time-dependent annihilation and creation operators. To do so, we introduce a pair of annihilation operators

$$\hat{d}_i(t) = \sum_{k=1,2}\left[\alpha_{ik}\hat{a}_k + \beta_{ik}\hat{a}_k^\dagger\right], \quad (i = 1, 2) \tag{33}$$

which satisfy the Liouville-von Neumann (LvN) equation for quantum invariants

$$i\frac{\partial}{\partial t}\hat{d}_i(t) + \left[\hat{d}_i(t), \hat{H}_\perp(t)\right] = 0. \tag{34}$$

And there are similar equations for the creation operators, $\hat{d}_i^\dagger(t)$, which are Hermitian conjugates of $\hat{d}_i(t)$. We further impose the equal-time commutation relations

$$\begin{aligned} \left[\hat{d}_i(t), \hat{d}_j^\dagger(t)\right] &= \delta_{ij}, \\ \left[\hat{d}_i(t), \hat{d}_j(t)\right] = \left[\hat{d}_i^\dagger(t), \hat{d}_j^\dagger(t)\right] &= 0. \end{aligned} \tag{35}$$

Assuming $B = B_0$ for $t \le t_0$ and $\omega(t_0) = \omega_0$, the initial conditions are

$$\begin{aligned} \alpha_{11}(t_0) &= \frac{1}{\sqrt{2}}, \quad \alpha_{12}(t_0) = \frac{i}{\sqrt{2}}, \\ \alpha_{21}(t_0) &= \frac{1}{\sqrt{2}}, \quad \alpha_{22}(t_0) = -\frac{i}{\sqrt{2}}, \end{aligned} \tag{36}$$

and

$$\beta_{ij}(t_0) = 0. \tag{37}$$

Factoring out the dynamical phases

$$\begin{aligned} \alpha_{ik} &= e^{\frac{i}{2}\int^t \Omega_+(t')dt'}\tilde{\alpha}_{ik}, \\ \beta_{ik} &= e^{-\frac{i}{2}\int^t \Omega_-(t')dt'}\tilde{\beta}_{ik}, \end{aligned} \tag{38}$$

the LvN equation gives a system of first order linear differential equations

$$\begin{pmatrix} \dot{\tilde{\alpha}}_{ik} \\ \dot{\tilde{\beta}}_{ik} \end{pmatrix} = \sum_{l=1,2} \begin{pmatrix} -\omega(t)\epsilon_{kl} & i\frac{\tilde{\Omega}_-(t)}{2}\delta_{kl} \\ -i\frac{\tilde{\Omega}_+(t)}{2}\delta_{kl} & -\omega(t)\epsilon_{kl} \end{pmatrix} \begin{pmatrix} \tilde{\alpha}_{il} \\ \tilde{\beta}_{il} \end{pmatrix}, \tag{39}$$

where

$$\tilde{\Omega}_-(t) = \Omega_-(t)e^{-i\int^t \Omega_+(t')dt'} \tag{40}$$

Without the ϵ_{kl}-term, the LvN equation is analogous to the spin system in a time-dependent magnetic field. Solving the LvN equation is essentially equivalent to solving time-dependent Schrödinger or Heisenberg equation. But the advantage of (quantum) invariants is straightforward construction of Landau states, coherent states, squeezed states, and their thermal counterparts.

Noting that $\hat{d}_1(t)$ is decoupled from $\hat{d}_2(t)$ and that $\hat{d}_1(t) = \hat{c}$ and $\hat{d}_2(t) = \hat{b}$ for $t \le t_0$, we may find the time-dependent Landau states

$$|n, l, t\rangle = \frac{(\hat{d}_1^\dagger(t))^n}{\sqrt{n!}} \frac{(\hat{d}_2^\dagger(t))^l}{\sqrt{l!}} |0, 0, t\rangle, \tag{41}$$

where the ground state is

$$\hat{d}_1(t)|0, 0, t\rangle = \hat{d}_d(t)|0, 0, t\rangle. \tag{42}$$

The exact Landau state is given by

$$|\Psi_{n,l}(t)\rangle = e^{i\int \langle n,n+l,t|i\frac{\partial}{\partial t}-\hat{H}(t)|n,n+l,t\rangle}|n,n+l,t\rangle. \quad (43)$$

Thus, the Landau states in a time-dependent magnetic field have the same infinite degeneracy ($l = 0, 1, 2, \cdots$) as those in a constant magnetic field.

V. Conclusion

The Landau levels for charges in magnetic fields are important in understanding condensed matter, such as quantum Hall effect, graphene, etc, and the equation of states of matter in neutron stars or magnetars. Magnetic fields can be used to control ions to the quantum limit in Paul traps, which may have applications to atomic physics and atomic interferometry. The quantum motion of charges in time-dependent magnetic fields is necessary for QED plasma.

In this paper, we have reviewed the Lewis-Riesenfeld quantum invariant for time-dependent oscillators. The time-dependent annihilation and creation operators, the quantum invariants, are expressed in terms of a complex solution to the equation for the oscillator. The planar Hamiltonian for a scalar charge in magnetic fields consists of two coupled oscillators. We have employed the oscillator representation for the planar Hamiltonian in a constant magnetic field and found the Landau levels and the infinite degeneracy corresponding to the angular momentum. We have introduced the time-dependent annihilation and creation operators, quantum invariants, for the planar Hamiltonian in time-dependent magnetic fields and expressed the governing equations for the operators.

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